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STATIC SOLUTIONS OF THE LEIPHOLZ PROBLEM

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Abstract—If the external loads on a bar remain in the original direction after buckling, the character of the problem is conservative. When these loads are tangentially-directed the problem is non-conservative and in this case the critical load must be calculated by the dynamic and by the static adjoint method.

In this paper the critical buckling load of the strut which is built in at the bottom and compressed by a distributed tangential load is calculated by the static adjoint method.

1. INTRODUCTION

A cantilever under distributed tangentially-directed load (Leipholz problem) is solved here by analogy to Reut's problem.

As is known, Reut's problem and Beck's problem are equivalent because the forces at the ends of the bar are identical in both cases (Bolotin, 1963; Feodosyev, 1977). By solving Beck's problem we obtain the adjoint critical load for Reut's problem and Leipholz's problem together.

As is known, the Leipholz problem is still unsolved statically. It was solved dynamically by Leipholz (1962) and Hauger (1966), and Peterson (1972) gives other examples in addition.

2. THE DIFFERENTIAL EQUATION OF A CANTILEVER BAR COMPRESSED BY DISTRIBUTED TANGENTIAL LOAD

We divide the distributed tangential load into vertical and horizontal components respectively [see Fig. 1(a)]:

$$q_{\mathrm{v}} \cong q \quad \text{and} \quad q_{\mathrm{h}} \cong qy', \tag{1}$$

and write the differential equation of the shearing forces

$$(EJy'')' + Qy' = -Q_{\mathrm{t}}, \tag{2}$$

where

$$Q = \int_0^x q \, \mathrm{d}x \tag{3}$$

is the load between any section and the free end, and

$$qy' = -Q_{\mathrm{t}}' \tag{4}$$

is a horizontal component of the load, and EJ is a bending rigidity of the beam.

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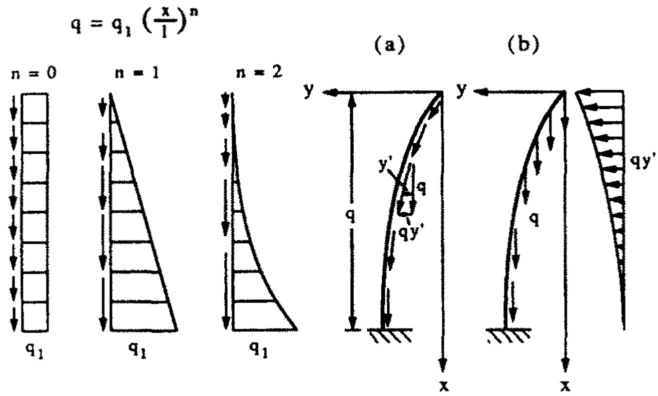

Fig. 1. The cantilever bar compressed by distributed tangential load (Leipholz problem).

By differentiating eqn (2) once and substituting expressions (3) and (4) into this equation we obtain a well-known differential equation

$$(EJy'')''+Qy''=0. \tag{5}$$

If a bar of variable cross-section is submitted to the action of a distributed tangentially-directed load, the differential equation of the deflected curve of the buckled bar can always be integrated by using Bessel functions, provided the distributed load and the flexural rigidity can be represented by the equations

$$q=q_1(x_1/l)^n, \tag{6}$$

$$EJ=EJ_1(x_1/l)^m, \tag{7}$$

where q_1 and J_1 are the intensity of load and the moment of inertia at the lower built-in end of the bar [see Figs 1(a) and (b)].

After substitution of (6) and (7) into eqn (5) we obtain

$$[EJ_1(x_1/l)^m y'']''+q_1 l(x_1/l)^{n+1}y''/(n+1)=0. \tag{8}$$

By substituting

$$(x_1/l)^m y''=z_1 \tag{9}$$

and

$$q_1 l/(n+1)=Q_1 \tag{10}$$

into the differential equation (8) we obtain a final form of the differential equation:

$$EJ_1(l/x_1)^{k+1}z_1''+Q_1 z_1=0, \tag{11}$$

where

$$k=n-m \tag{12}$$

and Q_1 is the total load on a bar.

The relevant boundary conditions are (see Fig. 1):

$$y''(0) = 0 \quad \text{and} \quad y'''(0) = 0, \tag{13}$$

or

$$z_1(0) = 0 \quad \text{and} \quad z_1'(0) = 0. \tag{14}$$

3. REUT'S PROBLEM [SEE BOLOTIN (1963)]

A bar fixed at one end has, at the free end a rigid disk to which is applied a force Q_1. The point of application of the force is always located on the axis.

The case of this loading [Fig. 2(b)] has become known as Reut's problem.

The differential equation (11) and the boundary conditions (14) also describe Reut's problem. Reut's problem and Beck's problem are equivalent, from the static point of view, because the forces at the ends of the bar are identical in both cases. The only difference is in the reference system for z and x [Fig. 2(c)]. If the new x axis is parallel to $z_1'(l)$, we obtain

$$z + z_1 = z'(0)x_1, \tag{15}$$

$$\mathrm{d}z_1/\mathrm{d}x_1 = z'(0) - \mathrm{d}z/\mathrm{d}x, \tag{16}$$

$$\mathrm{d}^2 z_1/\mathrm{d}x^2 = -\mathrm{d}^2 z/\mathrm{d}x^2, \tag{17}$$

and assuming that the displacements are small,

$$x_1 \cong x, \tag{18}$$

substituting eqns (15) and (17) into the differential equation (11) we obtain

$$EJ_1(l/x)^{k+1}z'' + Q_1 z - Q_1 z'(0)x = 0, \tag{19}$$

with the boundary conditions

$$z(0) = 0, \quad z'(l) = 0, \tag{20}$$

and

$$z'(0) = z'(0), \tag{20a}$$

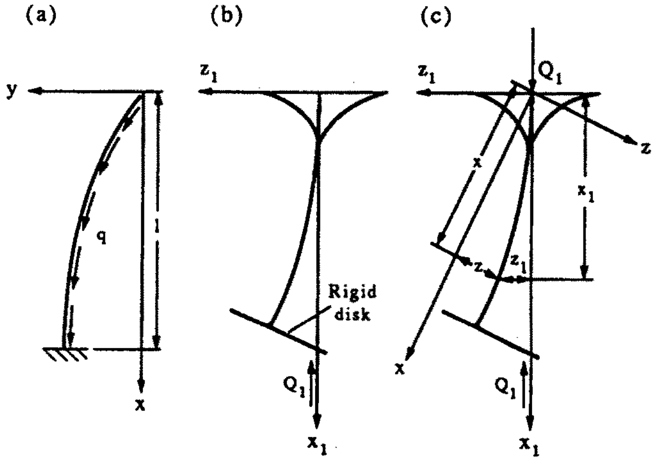

Fig. 2. Transformation of Leipholz's problem to Reut's problem and Reut's problem to Beck's problem.

which describe Beck's problem (the cantilever bar with its free end subjected to a follower force).

4. THE CANTILEVER BAR SUBJECTED TO FOLLOWER LOAD Q_1 WITH A LAG PARAMETER g

The proposed problem involves fundamentally new aspects of the static stability and cannot be solved by the conventional method. To illustrate the adjoint method of solution, let us consider the cantilever bar subjected at its free end to a follower load Q_1 with a lag parameter g, with a variable flexural rigidity and also loaded by a lateral concentrated force N (Fig. 3).

It was almost universally accepted that Beck's problem must be solved by the dynamic method. However this is not absolutely essential (Dmitriyuk, 1992).

At first we solve this problem by the conventional way. We divide the load Q_1 into vertical Q_1 and horizontal $gw'(0)Q_1$ components.

We set up the differential equation of the deflected axis of the bar:

$$EJ_1(l/x)^{k+1}w''+Q_1w-Q_1qw'(0)x = -Nx. \tag{21}$$

The general solution to this equation is:

$$w = A_1w_1(x)+A_2w_2(x)+gw'(0)x-Nx/Q_1, \tag{22}$$

where $w_1(x)$ and $w_2(x)$ are two independent solutions of the homogeneous equation

$$EJ_1(l/x)^{k+1}w''+Q_1w = 0.$$

The relevant boundary conditions are:

$$w(0) = 0, \quad w'(l) = 0, \tag{23}$$

and

$$w'(0) = w'(0). \tag{23a}$$

Determining the constants A_1, A_2 and $w'(0)$ from the boundary conditions (23) and the condition (23a), we obtain:

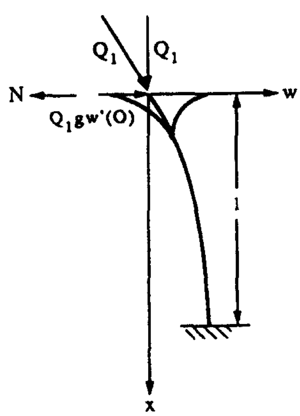

Fig. 3. The cantilever bar subjected at its free end to a follower force (Beck's problem).

$$w = Nl/Q_1\left(\begin{vmatrix} w_1(x) & w_2(x) \\ w_1(0) & w_2(0) \end{vmatrix} + D_2 x/l\right)\Big/ D, \tag{24}$$

or generally

$$w = f(x, Q_1)/q(Q_1, g) \tag{25}$$

and

$$w(l) = Nl(D_5 + D_2)/(Q_1 D), \tag{26}$$

$$w'(0) = Nl(D_2 - D_1)/(Q_1 D), \tag{27}$$

where

$$D = g(D_2 - D_1) - D_2,$$

or generally

$$D = q(Q_1, g) \tag{28}$$

and

$$D_1 = l\begin{vmatrix} w_1(0) & w_2(0) \\ w_1'(0) & w_1'(0) \end{vmatrix}, \quad D_2 = l\begin{vmatrix} w_1(0) & w_2(0) \\ w_1'(l) & w_2'(l) \end{vmatrix}, \quad D_5 = \begin{vmatrix} w_1(l) & w_2(l) \\ w_1(0) & w_2(0) \end{vmatrix}. \tag{29}$$

By equating the determinant of the system (28) to zero, we obtain the critical value of Q_1 for some range of the coefficient g:

$$g(D_2 - D_1) - D_2 = 0,$$

or generally

$$q(Q_1, g) = 0. \tag{30}$$

For $g = 1$ eqns (30) do not define the critical load Q_1, because D_1 is a particular case of the Wronskian determinant and cannot be zero.

In the literature there are a number of theories on the applicability of the conventional (Euler) method. However it is not difficult to obtain sufficient conditions so that the boundary-value problem has only real eigenvalues (Bolotin, 1963). It is well known that if a boundary-value problem is self-adjoint then all its eigenvalues are real.

The boundary-value problem is self-adjoint if by virtue of the boundary conditions the Green's integral

$$\int_0^l [wL(u) - uL(w)]\,\mathrm{d}x = 0 \tag{31}$$

vanishes for any choice of the functions w and u satisfying these conditions (Bolotin, 1963).

If

$$L[w - gw'(0)x] = [w - gw'(0)x]'' + Q_1[w - gw'(0)x]/[EJ_1(l/x)^{k+1}] \tag{32}$$

is the left-hand side of eqn (21) and

$$L[u - gu'(0)x] = [u - gu'(0)x]'' + Q_1[u - gu'(0)x]/[EJ_1(l/x)^{k+1}] \tag{33}$$

is the self-adjoint differential equation, from Green's integral (31) and the boundary conditions (23) we obtain:

$$g[u'(0)w(l) - w'(0)u(l)] = 0. \tag{34}$$

The problem is self-adjoint if:

$$(1) \quad g = 0 \quad (\text{"dead" load}), \tag{35}$$

$$(2) \quad w(l)/[gw'(0)] = u(l)/[gu'(0)] = d. \tag{36}$$

We define the expression (36) by substitution of eqns (26) and (27) for a lateral load N:

$$d/l = (D_5+D_2)/[g(D_2-D_1)]. \tag{37}$$

In the present paper, the scope of the static method is broadened since it is used in conjunction with Green's integral (31).

Now we rewrite the differential equation (21) in terms of u, and by substituting

$$gu'(0) = u(l)/d \tag{38}$$

from the condition (36), we obtain the self-adjoint equation

$$EJ_1(l/x)^{k+1}u''+Q_1u-Q_1u(l)x/l = -Nx. \tag{39}$$

The relevant boundary conditions are:

$$u(0) = 0, \quad u'(l) = 0, \tag{40}$$

and

$$u(l) = u(l). \tag{40a}$$

The differential equation (39) with the boundary conditions (40) describe the stability of a bar, with the variable flexural rigidity, loaded by force Q_1 through a fixed point F (see Fig. 4). Feodosyev (1977) solved this problem for a bar with a constant flexural rigidity.

The general solution is:

$$u = B_1w_1(x)+B_2w_2(x)+u(l)x/d-Nx/Q_1. \tag{41}$$

Determining the constants B_1, B_2 and $u(l)$ from the boundary conditions (40) and the condition (40a), we obtain:

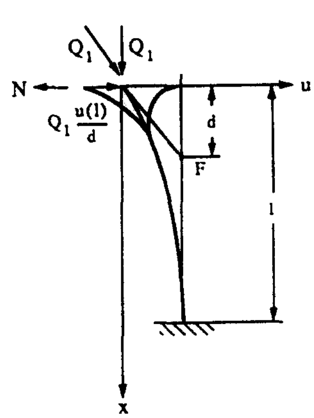

Fig. 4. The cantilever bar with load Q_1 through a fixed point F.

$$[(D_2+D_5)l/d-D_2]u = Nl\left(\begin{vmatrix} w_1(x) & w_2(x) \\ w_1(0) & w_2(0) \end{vmatrix}+D_2x/l\right)\Big/Q_1. \tag{42}$$

We rewrite eqn (42) for determination of the deflection

$$(D_2+D_5-dD_2/l)u = Nl\left(\begin{vmatrix} w_1(x) & w_2(x) \\ w_1(0) & w_2(0) \end{vmatrix}+D_2x/l\right)d/(Q_1l). \tag{43}$$

The characteristic equation is:

$$D\# = (D_2+D_5)l/d-D_2. \tag{44}$$

We rewrite eqn (44) for determination of the critical load

$$D_2+D_5-D_2d/l = 0. \tag{45}$$

After substitution of eqn (37) into eqn (43), we obtain the adjoint expression of deflection for the concentrated load N

$$\left\{u-\left(\begin{vmatrix} w_1(x) & w_2(x) \\ w_1(0) & w_2(0) \end{vmatrix}+D_2x/l\right)Nl/Q_1[g(D_2-D_1)-D_2]\right\}(D_2+D_5)/[g(D_2-D_1)] = 0 \tag{46}$$

or generally

$$[u-f(x,Q_1)/q(Q_1,g)]r(Q_1) = 0. \tag{47}$$

These equations are expanded into two expressions. In discussing the solutions of eqns (46) and (47) let us begin with two cases:

(1) As long as the second expression is not equal to zero the first expression will be equal to zero and will give a deflection identical to the one found by normal static consideration [see eqns (24) and (25)].

(2) If the second expression is equal to zero we can determine the critical load.

Now, because the first expression must not be zero, the deflection is unstable.

Now, from the characteristic equation (45) after substitution of eqn (37), we obtain an expression for the determination of the critical load:

$$(D_2+D_5)[g(D_2-D_1)-D_2]/[g(D_2-D_1)] = 0 \tag{48}$$

or generally

$$r(Q_1)q(Q_1,g) = 0. \tag{49}$$

Equations (48) and (49) are also expanded into two expressions. From the first expression we obtain the adjoint critical load:

$$D_2+D_5 = 0, \tag{50}$$

$$r(Q_1) = 0. \tag{51}$$

The second expression gives us a critical load exactly like the one we found using normal static consideration [see eqn (30)].

It should be noted that from eqn (47) we can obtain another equivalent form:

$$r(Q_1)u = f(x, Q_1)r(Q_1)/q(Q_1, g), \tag{52}$$

or finally

$$u = \frac{f(x, Q_1)}{q(Q_1, g)} \frac{r(Q_1)}{r(Q_1)}. \tag{53}$$

If we cancel out the factor $r(Q_1)$ from the numerator and the denominator, we lose the adjoint solution. This means that if we obtain the equivalent form (53) and go to the adjoint form (47) instead of cancelling, we will obtain the adjoint solution (Dmitriyuk, 1992).

Equation (50) gives the critical load for Beck's, Reut's ($g = 1$) and Leipholz's problems ($g = 1$) for a lateral concentrated load N.

5. THE CRITICAL LOAD OF A BAR WITH A CONSTANT MOMENT OF INERTIA ($m = 0$; $k = n$) WHICH IS BUILT IN AT THE BOTTOM AND COMPRESSED BY A DISTRIBUTED TANGENTIAL LOAD [EQN (50)]

$$\begin{aligned} w_1 &= x^{1/2}J_{1/(n+3)}(2ax^{(n+3)/2}/(n+3)), \\ w_2 &= x^{1/2}J_{-(1/(n+3))}(2ax^{(n+3)/2}/(n+3)), \\ w'_1 &= ax^{(n+2)/2}J_{-((n+2)/(n+3))}(2ax^{(n+3)/2}/(n-3)), \\ w'_2 &= -ax^{(n+2)/2}J_{-((n+2)/(n+3))}(2ax^{(n+3)/2}/(n+3)), \end{aligned} \tag{54}$$

consequently

$$D_2 = -\frac{(n+3)^{1/(n+3)}}{a^{1/(n+3)}\Gamma(\frac{n+2}{n+3})} al^{(n+4)/2}J_{-((n+2)/(n+3))}(2al^{(n+3)/2}/(n+3)), \tag{55}$$

$$D_5 = \frac{(n+3)^{1/(n+3)}}{a^{1/(n+3)}\Gamma(\frac{n+2}{n+3})} l^{1/2}J_{1/(n+3)}(2al^{(n+3)/2}/(n+3)), \tag{56}$$

where from the differential equation (11)

$$a^2 = Q_1/EJ_1l^{n+1} = q_1/[EJ_1l^n(n+1)], \tag{57}$$

and finally eqn (50) gives

$$D_2 + D_5 = \frac{(n+3)^{1/(n+3)}l^{1/2}}{a^{1/(n+3)}\Gamma(\frac{n+2}{n+3})}\left(J_{1/(n+3)}(X) - \frac{n+3}{2}XJ_{-((n+2)/(n+3))}(X)\right) = 0, \tag{58}$$

where

$$X = 2al^{(n+3)/2}/(n+3). \tag{59}$$

The results are tabulated as follows:

	The adjoint method			The dynamic method
$n = -1$	$X = al \cong 4.493$	$Q_1 \cong 20.19EJ_1/l^2$,	$q_1 = 0$	$Q_1 \cong 20.05EJ_1/l^2$ (Beck, 1952)
$n = 0$	$X = 2al^{3/2}/3 \cong 4.27$	$Q_1 \cong 41EJ_1/l^2$,	$q_1 \cong 41EJ_1/l^3$	$q_1 \cong 40.7EJ_1/l^3$ (Leipholz, 1962)
$n = 1$	$X = al^2/2 \cong 4.16$	$Q_1 \cong 69.2EJ_1/l^2$,	$q_1 \cong 138.4EJ_1/l^3$	$q_1 \cong 158.2EJ_1/l^3$ (Hauger, 1966) $q_1 \cong 149.82EJ_1/l^3$ (Petersen, 1972)
$n = 2$	$X = 2al^{5/2}/5 \approx 4.09$	$Q_1 \approx 105EJ_1/l^2$,	$q_1 \approx 314EJ_1/l^3$	
$n = 3$	$X = al^3/3 \approx 4.06$	$Q_1 \approx 148EJ_1/l^2$,	$q_1 \approx 593EJ_1/l^3$	

6. THE CRITICAL LOAD OF A SAMPLE SUPPORTED BAR WITH CONSTANT MOMENT OF INERTIA ($m = 0$; $k = n$) AND COMPRESSED BY DISTRIBUTED TANGENTIAL LOAD

This case we can solve by the conventional (Euler) method.

The differential equation for this case is eqn (11) with boundary conditions (see Fig. 5):

$$z_1(0) = 0 \quad \text{and} \quad z_1(l) = 0. \tag{60}$$

The general solution of eqn (11) is

$$z_1(x) = C_1 w_1(x) + C_2 w_2(x), \tag{61}$$

which gives

$$D_5 = 0 \quad \text{or} \quad J_{1/(n+3)}(X) = 0. \tag{62}$$

We draw up the following table:

$n = -1$	$X = 3.14159$	$q = 0,\ \ Q \cong 9.8696EJ_1/l^2$	
$n = 0$	$X \cong 2.9026$	$q \cong 18.96EJ_1/l^3$	$q \cong 18.96EJ_1/l^3$ (Leipholz, 1962)
$n = 1$	$X \cong 2.7809$	$q \cong 61.87EJ_1/l^3$	$q \cong 61.9EJ_1/l^3$ (Petersen, 1972) $q \cong 62.28EJ_1/l^3$ (Hauger, 1966)
$n = 2$	$X \cong 2.7070$	$q \cong 137.4EJ_1/l^3$	
$n = 3$	$X \cong 2.6575$	$q \cong 245.24EJ_1/l^3$	

For a better understanding of the meaning of the adjoint critical load, the critical load for the phenomenon of reversal of deflection of a bar with elastically built-in ends is calculated.

7. THE USE OF BEAM-COLUMN THEORY IN CALCULATING CRITICAL LOADS OF THE PHENOMENON OF REVERSAL OF DEFLECTION

Let us consider a bar with elastically built-in ends. An example of such end conditions is presented in Fig. 6. A laterally loaded beam AB is rigidly connected to vertical bars at A and B and is compressed axially by the forces P. If Θ_a and Θ_b are the angles of rotation of the ends, there will be couples M_a and M_b at the end of the beam (see Fig. 6) which can be expressed in the form:

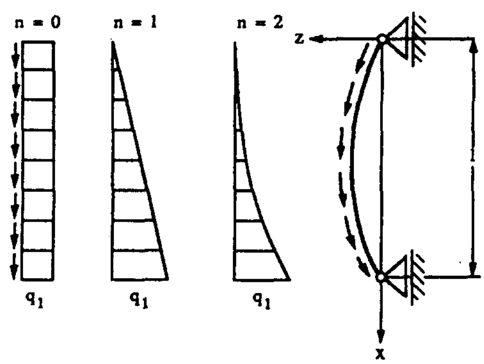

Fig. 5. A simply-supported bar compressed by a distributed tangential load.

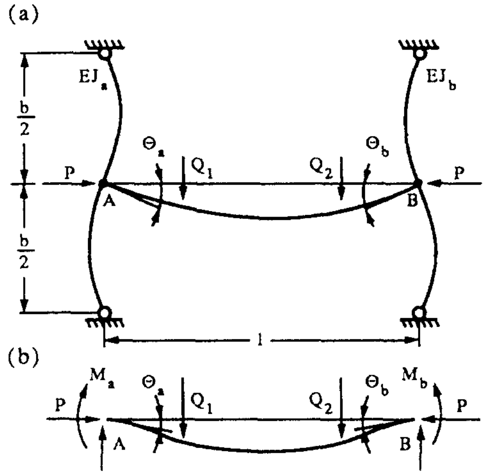

Fig. 6. A bar with elastically built-in ends.

$$M_a = -\alpha\Theta_a, \quad M_b = -\beta\Theta_b. \tag{63}$$

The moments and angles of rotation are taken to be positive in the directions shown in Fig. 6. The factors α and β are called coefficients of end restraint.

The angles Θ_a and Θ_b can now be determined from a consideration of the bending of the bar AB. We obtain

$$\Theta_a = \Theta_{0a} + \frac{M_a l}{3EJ}\Psi(u) + \frac{M_b l}{6EJ}\Phi(u),$$

$$\Theta_a = \Theta_{0b} + \frac{M_b l}{3EJ}\Psi(u) + \frac{M_a l}{6EJ}\Phi(u). \tag{64}$$

Finally, from eqn (63) and eqn (64), the following equations for determining the moments at the ends are obtained:

$$-\frac{M_a}{\alpha} = \Theta_{0a} + \frac{M_a l}{3EJ}\Psi(u) + \frac{M_b l}{6EJ}\Phi(u),$$

$$-\frac{M_a}{\beta} = \Theta_{0b} + \frac{M_b l}{3EJ}\Psi(u) + \frac{M_a l}{6EJ}\Phi(u), \tag{65}$$

where Θ_{0a} and Θ_{0b} represent the angles of rotation at the ends due to lateral load only, EJ is the bending rigidity of the bar AB and $\Psi(u)$ and $\Phi(u)$ are known as Berry functions:

$$\Psi(u) = \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right), \tag{66}$$

$$\Phi(u) = \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right). \tag{67}$$

Solving eqns (65) for the moment M_a gives [see Timoshenko and Gere (1961)]

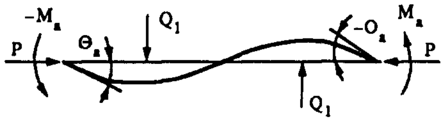

Fig. 7. The antisymmetrical loading on the symmetrically supported bar *AB*.

$$M_a = \frac{-\Theta_{0a}\left[\frac{1}{\beta}+\frac{l}{3EJ}\Psi(u)\right]+\Theta_{0b}\left[\frac{l}{6EJ}\Phi(u)\right]}{\left[\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)\right]\left[\frac{1}{\beta}+\frac{l}{3EJ}\Psi(u)\right]-\left[\frac{l}{6EJ}\Phi(u)\right]^2}. \tag{68}$$

If the loading on the symmetrically supported bar is antisymmetrical (Fig. 7) then we have

$$\alpha = \beta, \quad \Theta_{0a} = -\Theta_{0b}, \quad M_a = -M_b, \tag{69}$$

and the moment is determined by the equation

$$M_a = -\frac{\Theta_{0a}\left[\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)+\frac{l}{6EJ}\Phi(u)\right]}{\left[\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)+\frac{l}{6EJ}\Phi(u)\right]\left[\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)-\frac{l}{6EJ}\Phi(u)\right]} \tag{70}$$

in an equivalent form (53).

If we cancel out the factor

$$\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)+\frac{l}{6EJ}\Phi(u)$$

from the numerator and denominator, we lose the adjoint solution (the symmetrical one).

We will go to the adjoint form [eqn (47)]:

$$\left[\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)+\frac{l}{6EJ}\Phi(u)\right]\left[M_a+\frac{\Theta_{0a}}{\frac{1}{\alpha}+\frac{l}{3EJ}\Psi(u)-\frac{l}{6EJ}\Phi(u)}\right] = 0. \tag{71}$$

Setting the denominator of the second factor equal to zero, we obtain the equation for the critical load, which corresponds to the antisymmetric buckling shape. Setting the first factor equal to zero, we obtain the missing equation for the critical load which corresponds to the symmetrical buckling shape [see Timoshenko and Gere (1961)], where

$$u = (P/EJ)^{1/2} l/2. \tag{72}$$

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